GIBBS-LIKE MEASURE FOR SPECTRUM OF A CLASS OF ONE-DIMENSIONAL SCHRÖDINGER OPERATOR WITH STURM POTENTIALS

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Abstract. Let $\alpha \in (0,1)$ be an irrational, and $[0; a_1, a_2, \cdots]$ the continued fraction expansion of α. Let $H_{\alpha,V}$ be the one-dimensional Schrödinger operator with Sturm potential of frequency α. Suppose the potential strength V is large enough and $(a_i)_{i\geq 1}$ is bounded. We prove that the spectral generating bands possess properties of bounded distortion, bounded covariation and there exists Gibbs-like measure on the spectrum $\sigma(H_{\alpha,V})$. As an application, we prove that

$$\dim_H \sigma(H_{\alpha,V}) = s_*, \quad \overline{\dim}_B \sigma(H_{\alpha,V}) = s^*,$$

where s_* and s^* are lower and upper pre-dimensions.
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1. Introduction

Since the discovery of quasi-crystal by Schechtman et al. ([13]), the one dimensional discrete Schrödinger operators with Sturm potentials have largely been studied, see [1, 12, 14] and references therein. The discrete Schrödinger operator acting on $l^2(\mathbb{Z})$ is defined as follows: for any $\psi = \{\psi_n\}_{n\in\mathbb{Z}} \in l^2(\mathbb{Z})$,

$$(H_{\alpha,V}\psi)_n = \psi_{n-1} + \psi_{n+1} + v_n\psi_n, \ \forall n \in \mathbb{Z}. \tag{1.1}$$

The potential $(v_n)_{n\in\mathbb{Z}}$ we discuss in this paper is the Sturm potential, i.e.,

$$v_n = V\chi_{[1-\alpha,1)}(n\alpha + \phi \mod 1), \quad \forall n \in \mathbb{Z}, \tag{1.2}$$

where $\alpha \in (0,1)$ is an irrational, and is called frequency, $V > 0$ is called potential strength or coupling, $\phi \in [0,1)$ is called phase. We will study the structure of the spectrum of the operator which we denote by $\sigma(H_{\alpha,V})$, in particular the fractal dimensions of $\sigma(H_{\alpha,V})$. It is known that $\sigma(H_{\alpha,V})$ is independent of phase ϕ, we set $\phi = 0$.

It is proved by Bellissard, Iochum, Scoppola and Testart ([1], 1989) that $\sigma(H_{\alpha,V})$ is a Cantor set of Lebesgue measure zero. (On the other direction, in stead of Sturm potential, some authors considered the primitive substitutive potential, it is proved that in this case,

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the spectrum is also a Cantor set of Lebesgue measure zero. For more details, we refer to [6],[8].)

Since then, whether the Hausdorff dimension of $\sigma(H_{\alpha,V})$ is strictly less than 1 and strictly greater than 0 have absorbed a lot of attentions. Raymond [12](1997) studied this problem under the restriction $V>4$. Connected with the continued fraction expansion of α, he exhibited an interesting recurrent structure of the spectrums. And for $\alpha=\frac{\sqrt{5}-1}{2}$, i.e., the golden mean, he gave an upper bound of the Hausdorff dimension of the corresponding spectrum, which is strictly less than 1.

Damanik, Killip and Lenz [2](2000) proved that if α has bounded density (this means if $[0;a_1,a_2,\cdots]$ is the continued fraction expansion of α, then $\limsup_{k\to\infty}\frac{1}{k}\sum_{i=1}^k a_i<\infty$), then the Hausdorff dimension of the spectral measure of $H_{\alpha,V}$ is strictly greater than 0. Since the spectral measure is supported by the spectrum $\sigma(H_{\alpha,V})$, the Hausdorff dimension of the spectrum has also strictly positive lower bound.

To estimate fractal dimensions of the spectrum of $H_{\alpha,V}$, one of the key steps is to estimate the length of spectral generating bands. Raymond [12] has treated the case of frequency α being golden mean with $V>4$. Based on the Raymond's method, for all irrational frequency and $V>20$, Liu and Wen ([9],[10]) established multi-type Moran construction among different spectral generating bands, developed a very fine estimating technique for the length of the bands of different orders of the spectrum $\sigma(H_{\alpha,V})$, and generalized some techniques analogous to the studies of Moran structure in [4, 5], they proved the following result.

Theorem A1. *Let $\alpha=[0;a_1,a_2,\cdots]$ be irrational and*

$$K=\liminf_{k\to\infty}(a_1a_2\cdots a_k)^{1/k}.$$

Let $V>20$ and $t_1=\frac{3}{V-8}$, $t_2=\frac{1}{4(V+8)}$.

(1) *If $K<\infty$, then*

$$\max\{\frac{\ln 2}{10\ln 2-3\ln t_2},\frac{\ln K-\ln 3}{\ln K-\ln(t_2/3)}\}\le \dim_H\,\sigma(H_{\alpha,V})\le\frac{2\ln K+\ln 3}{2\ln K-\ln t_1};$$

(2) *If $K=\infty$, then*

$$\dim_H\,\sigma(H_{\alpha,V})=1.$$

Note that this theorem implies that if $K<\infty$, then $\dim_H(\sigma(H_{\alpha,V}))$ tends to 0 when V tends to infinity.

Damanik, Embree, Gorodetski, and Tcheremchantsev ([3]) proved that, for golden mean α,

$$\lim_{V\to\infty}(\log V)\,\overline{\dim}_B\sigma(H_{\alpha,V})=\log(1+\sqrt{2}),$$

and found that $\dim_H\,\sigma(H_{\alpha,V})=\overline{\dim}_B\,\sigma(H_{\alpha,V})$ by applying dimensional theory of dynamical system.

Liu, Peyrière, Wen[7] extended their results to case of $\alpha=[0;a_1,a_2,\cdots]$ with $(a_n)_{n\ge1}$ bounded. They proved that, for pre-dimensions $0\le s_*\le s^*\le 1$ (which will be defined in §2),

$$\dim_H\sigma(H_{\alpha,V})\le s_*,\quad \overline{\dim}_B\sigma(H_{\alpha,V})\ge s^*,$$

and

$$\lim_{V\to\infty} s_* \log V = -\log f_*(\alpha), \quad \lim_{V\to\infty} s^* \log V = -\log f^*(\alpha),$$

where $f_*(\beta)$ and $f^*(\beta)$ are the positive roots of the equations

$$\begin{aligned} &\liminf_{n\to\infty} \|\mathbf{R}_1(x)\mathbf{R}_2(x)\cdots\mathbf{R}_n(x)\|^{1/n} = 1 \\ &\limsup_{n\to\infty} \|\mathbf{R}_1(x)\mathbf{R}_2(x)\cdots\mathbf{R}_n(x)\|^{1/n} = 1, \end{aligned}$$

and for any $0 < x \le 1$ and $n \ge 1$,

$$\mathbf{R}_n(x) := \begin{pmatrix} 0 & x^{(a_n-1)} & 0 \\ (a_n+1)x & 0 & a_n x \\ a_n x & 0 & (a_n-1)x \end{pmatrix}.$$

If $\alpha = [0; 1, 1, \cdots]$, then $f_*(\alpha) = f^*(\alpha) = (1+\sqrt{2})^{-1}$, which is the positive root of

$$\det\left(\begin{bmatrix} 0 & 1 & 0 \\ 2x & 0 & x \\ x & 0 & 0 \end{bmatrix} - I \right) = -1 + 2x + x^2 = 0.$$

They also show that there are frequencies α with $f_*(\alpha) < f^*(\alpha)$.

In this paper, we will consider the general formula of the dimensions of the spectrum for the case $(a_n)_{n\ge1}$ being bounded. For this aim, we establish first the properties of bounded variation, bounded covariation for spectral generating bands, then prove the existence of Gibbs-like measures for spectrum, finally we give a general result of the Hausdorff dimension and upper box dimension of the spectrum, that is,

$$\dim_H \sigma(H_{\alpha,V}) = s_*, \quad \overline{\dim}_B \sigma(H_{\alpha,V}) = s^*.$$

The remainder of the paper will be organized as follows: in Section 2, we introduce spectral structure and state the main results of the paper; Section 3 will be devoted to the proofs of these results.

2. Spectral structure

We discuss first some facts on the structure of $\sigma(H_{\alpha,V})$.

Let $\alpha = [a_1, a_2, \cdots, a_i, \cdots] \in (0,1)$ be an irrational, let $p_k/q_k (k > 0)$ be the k-th asymptotic fraction of α given by:

$$\begin{aligned} p_{-1} &= 1, \quad p_0 = 0, \quad p_{k+1} = a_{k+1}p_k + p_{k-1},\ k \ge 0, \\ q_{-1} &= 0, \quad q_0 = 1, \quad q_{k+1} = a_{k+1}q_k + q_{k-1},\ k \ge 0. \end{aligned}$$

Let $k \ge 1$ and $x \in \mathbb{R}$, the transfer matrix $M_k(x)$ over q_k sites is defined by

$$\mathbf{M}_k(x) := \begin{bmatrix} x - v_{q_k} & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x - v_{q_k-1} & -1 \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} x - v_2 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x - v_1 & -1 \\ 1 & 0 \end{bmatrix},$$

where v_n is defined in (1.2) and by convention, take

$$\mathbf{M}_{-1}(x) = \begin{bmatrix} 1 & -V \\ 0 & 1 \end{bmatrix}, \quad \mathbf{M}_0(x) = \begin{bmatrix} x & -1 \\ 1 & 0 \end{bmatrix}.$$

For $k \ge 0$, $p \ge -1$, let $t_{(k,p)}(x) = \operatorname{tr} \mathbf{M}_{k-1}(x)\mathbf{M}_k^p(x)$ and $\sigma_{(k,p)} = \{x \in \mathbb{R} : |t_{(k,p)}(x)| \le 2\}$, where trM stands for the trace of the matrix M.

With these notations, we collect some known facts that will be used later, for more details, we refer to [1, 12, 14, 15].

(A) Renormalization relation. For any $k \geq 0$

$$\mathbf{M}_{k+1}(x) = \mathbf{M}_{k-1}(x)(\mathbf{M}_k(x))^{a_{k+1}}, \tag{2.1}$$

so, $t_{(k+1,0)} = t_{(k,a_k)}$, $t_{(k,-1)} = t_{(k-1,a_k-1)}$.

(B) Structure of $\sigma_{(k,p)}(k \geq 0, p \geq -1)$.
For $V > 0$, $\sigma_{(k,p)}$ is made out of $\deg t_{(k,p)}$ separated closed intervals.

(C) Trace relation.
By defining $\Lambda(x, y, z) = x^2 + y^2 + z^2 - xyz - 4$,

$$\Lambda(t_{(k+1,0)}, t_{(k,p)}, t_{(k,p+1)}) = V^2. \tag{2.2}$$

Thus for any $k \in \mathbb{N}$, $p \geq 0$ and $V > 4$,

$$\sigma_{(k+1,0)} \cap \sigma_{(k,p)} \cap \sigma_{(k,p-1)} = \emptyset. \tag{2.3}$$

(D) Covering property.
For any $k \geq 0$, $p \geq -1$,

$$\sigma_{(k,p+1)} \subset \sigma_{(k+1,0)} \cup \sigma_{(k,p)},$$

then

$$(\sigma_{(k+2,0)} \cup \sigma_{(k+1,0)}) \subset (\sigma_{(k+1,0)} \cup \sigma_{(k,0)}).$$

Moreover

$$\sigma(H_{\alpha,V}) = \bigcap_{k\geq 0}(\sigma_{(k+1,0)} \cup \sigma_{(k,0)}).$$

We call the constructive intervals of $\sigma_{(k,p)}$ the *bands*. When we discuss only one of these bands, we often denote it as $B_{(k,p)}$. Property (B) also implies $t_{(k,p)}(x)$ is monotone on $B_{(k,p)}$, and

$$t_{(k,p)}(B_{(k,p)}) = [-2, 2],$$

we call $t_{(k,p)}$ the *generating polynomial* of $B_{(k,p)}$.

Definition 1. ([12, 9]) *For $V > 4$, $k \geq 0$, we define three types of bands as follows:*
(k, I)-type band: a band of $\sigma_{(k,1)}$ contained in a band of $\sigma_{(k,0)}$;
(k, II)-type band: a band of $\sigma_{(k+1,0)}$ contained in a band of $\sigma_{(k,-1)}$;
(k, III)-type band: a band of $\sigma_{(k+1,0)}$ contained in a band of $\sigma_{(k,0)}$.

The three kinds of types of bands are well defined([12]), and we call these bands *spectral generating bands of order* k (the type I band is called the type I gap in [12]). Note that for order 0, there is only one $(0, \mathrm{I})$-type band $\sigma_{(0,1)} = [V-2, V+2]$ (the corresponding generating polynomial is $t_{(0,1)} = x - V$), and only one $(0, \mathrm{III})$ type band $\sigma_{(1,0)} = [-2, 2]$ (the corresponding generating polynomial is $t_{(1,0)} = x$). They are contained in $\sigma_{(0,0)} = (-\infty, +\infty)$ with corresponding generating polynomial $t_{(0,0)} \equiv 2$. For the convenience, we call $\sigma_{(0,0)}$ the spectral generating band of order -1.

For any $k \geq -1$, denote by $\mathscr{G}_k$ the set of all spectral generating bands of order k. By the properties (A),(B),(C) and (D), for any $k \geq 0$, we have

- $(\sigma_{(k+2,0)} \cup \sigma_{(k+1,0)}) \subset \bigcup_{B \in \mathscr{G}_k} B \subset (\sigma_{(k+1,0)} \cup \sigma_{(k,0)})$, and then
$$\sigma(H_{\alpha,V}) = \bigcap_{k \geq -1} \bigcup_{B \in \mathscr{G}_k} B;$$
- any $(k+1, I)$ or $(k+1, III)$-type band is contained in a (k, II) or (k, III)-type band; any $(k+1, II)$-type band is contained in a (k, I)-type band;
- any (k, II) or (k, III) do not contain any $(k+1, II)$-type band; any (k, I)-type band contain neither $(k+1, I)$ nor $(k+1, III)$-type band.

To show that one band of order k contains how many bands of order $k+1$, we introduce Chebischev polynomial $S_p(x)$, which is defined by

$$\begin{aligned} &S_0(x) \equiv 0, \quad S_1(x) \equiv 1, \\ &S_{p+1}(x) = xS_p(x) - S_{p-1}(x), \quad p \geq 1. \end{aligned}$$

By induction we see that

$$S_p(2\cos\theta) = \frac{\sin p\theta}{\sin\theta}, \quad \theta \in [0, \pi]. \tag{2.4}$$

Our study focus on the following three formulas according to the types of the band(see [1, 12, 9]):

$$t_{(k,p)} = t_{(k,0)} S_{p+1}(t_{(k+1,0)}) - t_{(k,-1)} S_p(t_{(k+1,0)}). \tag{2.5}$$

$$t_{(k,p+1)} = t_{(k,1)} S_{p+1}(t_{(k+1,0)}) - t_{(k,0)} S_p(t_{(k+1,0)}). \tag{2.6}$$

$$t_{(k,p+1)} = t_{(k+1,0)} t_{(k,p)} - t_{(k,p-1)}. \tag{2.7}$$

These three formulas can be obtained by the following way: let A be a 2×2 matrix with $|A| = 1$, then by Caylay-Hamilton Theorem $A^2 - (\operatorname{tr} A)A + I = 0$, and hence, for any $n > 1$,

$$\begin{aligned} A^n &= S_n(\operatorname{tr} A)\, A - S_{n-1}(\operatorname{tr} A)\, I \\ &= S_{n+1}(\operatorname{tr} A)\, I - S_n(\operatorname{tr} A)\, A^{-1}. \end{aligned}$$

Then take the trace in the both sides and by the definitions of $t_{(k,p)}$, the three formulas come.

Now consider the equation

$$\Lambda(x, y, z) = V^2,$$

then

$$z_{\pm}(x, y, V) = \frac{xy}{2} \pm \frac{1}{2}\sqrt{4V^2 + (4 - x^2)(4 - y^2)}. \tag{2.8}$$

For two branches $z = z_+$ or $z = z_-$, let

$$\begin{aligned} &z_1(x,y,V) := \tfrac{\partial z(x,y,V)}{\partial x},\ z_2(x,y,V) := \tfrac{\partial z(x,y,V)}{\partial y},\ z_{11}(x,y,V) := \tfrac{\partial^2 z(x,y,V)}{\partial x \partial x}, \\ &z_{12}(x,y,V) := \tfrac{\partial^2 z(x,y,V)}{\partial x \partial y},\ z_{21}(x,y,V) := \tfrac{\partial^2 z(x,y,V)}{\partial y \partial x},\ z_{22}(x,y,V) := \tfrac{\partial^2 z(x,y,V)}{\partial y \partial y}. \end{aligned}$$

For any $|x| \leq 2$, $|y| \leq 2$, and $V > 4$,

$$\begin{aligned} &V - 2 \leq |z_{\pm}(x,y,V)| \leq V + 2, \quad |z_1(x,y,V)| \leq 1,\ |z_2(x,y,V)| \leq 1, \\ &|z_{11}(x,y,V)| \leq 1,\ |z_{12}(x,y,V)| \leq 1,\ |z_{21}(x,y,V)| \leq 1,\ |z_{22}(x,y,V)| \leq 1. \end{aligned} \tag{2.9}$$

In the previous papers [12, 9], the authors have estimated the derivatives of the generating polynomials and the number of the bands of different types through the formulas (2.5), (2.6) and (2.7). Since the present situation are much more complicated, for treating the relations among different types of bands, we introduce the notion of ladder as follows.

For any $n > k \geq -1$, let

$$B_n \subseteq B_{n-1} \subseteq \cdots \subseteq B_k,$$

be a sequence of spectral generating bands from order n to k. We call the sequence $(B_i)_{i=k}^n$ an *initial ladder*, and the bands $B_i (k \leq i \leq n)$ are called initial rungs. Now we are going to modify the initial ladder by the following way: for any $i (k \leq i \leq n-1)$,

- if B_i is of (i, I)-type with $a_{i+1} = 1$, we delete the rung B_{i+1}(in this case B_{i+1} must be $(i+1, II)$-type, then $t_{(i+2,0)} = t_{(i,1)}$ and $t_{(i+1,-1)} = t_{(i,0)}$ implies $B_{i+1} = B_i$);
- if B_i is of (i, I)-type with $a_{i+1} = 2$, we change nothing;
- if B_i is of (i, I)-type with $a_{i+1} > 2$, we add rungs $(B_{(i,p)})_{p=2}^{a_{i+1}-1}$ between B_i and B_{i+1} :

$$B_{i+1} = B_{(i,a_i)} \subset B_{(i,a_i-1)} \subset \cdots \subset B_{(i,2)} \subset B_{(i,1)} = B_i;$$

- if B_i is of (i, II) or (i, III)-type, we change nothing.

We get by this way a unique modified ladder which we relabel as

$$B_n = \hat{B}_m \subset \cdots \subset \hat{B}_1 \subset \hat{B}_0 = B_k.$$

We call $(\hat{B}_i)_{i=0}^m$ the *modified ladder*, and we denote the corresponding generating polynomials by $(\hat{h}_i)_{i=0}^m$. Note that any two consecutive initial rungs can not be of type I simultaneously, so the length of the modified ladder is larger than $[(n-k)/2]$.

Although we do not define type for the bands of order -1, note that the bands of order 0 are either of type I or of type III, we can view $B_{-1} = \sigma_{(0,0)}$ as a band of type II or III, thus we need not add rungs between B_0 and B_{-1}.

Let $(\hat{B}_i)_{i=0}^m$ be a modified ladder and $(\hat{h}_i)_{i=0}^m$ the correspondent generating polynomials. Using the notion of ladders, we can unify the formulas (2.5)–(2.7) by one single formula. To see this, for any $-1 < i < m$, note that $\hat{B}_i$ may be in one of the following four situation: type I, II, III of a order $k \geq 0$ and an added rung of a order $k \geq 0$, and we distinguish them further into the following three cases.

Case 1) $\hat{B}_i$ is of (k, II)-type.

In this case, $\hat{h}_i = t_{(k+1,0)}$, $\hat{h}_{i+1} = t_{(k,p)}$ for some $p \geq 1$, $\hat{h}_{i-1} = t_{(k,-1)} = t_{(k-1,a_k-1)}$ (note that $\hat{B}_{i-1}$ is an added rung if $a_k > 2$, this is also an advantage to apply modified ladder). We have

$$\begin{aligned} \hat{h}_{i+1} = t_{(k,p)} &= t_{(k,0)} S_{p+1}(t_{(k+1,0)}) - t_{(k,-1)} S_p(t_{(k+1,0)}) \\ &= t_{(k,0)} S_{p+1}(\hat{h}_i) - \hat{h}_{i-1} S_p(\hat{h}_i) \\ t_{(k,0)} &= z_\pm(\hat{h}_i, \hat{h}_{i-1}, V). \end{aligned}$$

Case 2) $\hat{B}_i$ is of (k, III)-type.

In this case, $\hat{h}_i = t_{(k+1,0)}$, $\hat{h}_{i-1} = t_{(k,0)}$, $\hat{h}_{i+1} = t_{(k,p+1)}$ for some $p \geq 0$. We have

$$\begin{aligned} \hat{h}_{i+1} = t_{(k,p+1)} &= t_{(k,1)} S_{p+1}(t_{(k+1,0)}) - t_{(k,0)} S_p(t_{(k+1,0)}) \\ &= t_{(k,1)} S_{p+1}(\hat{h}_i) - \hat{h}_{i-1} S_p(\hat{h}_i) \\ t_{(k,1)} &= z_\pm(\hat{h}_i, \hat{h}_{i-1}, V), \end{aligned}$$

Case 3) $\hat{B}_i$ is of (k, I)-type, or, an added rung in order k.

In this case, there exists $1 \leq p \leq a_{k+1}$ such that $\hat{h}_i = t_{(k,p)}$, $\hat{h}_{i+1} = t_{(k,p+1)}$, $\hat{h}_{i-1} = t_{(k,p-1)}$, and

$$\begin{aligned} \hat{h}_{i+1} = t_{(k,p+1)} &= t_{(k+1,0)} t_{(k,p)} - t_{(k,p-1)} \\ &= t_{(k+1,0)} S_2(\hat{h}_i) - \hat{h}_{i-1} S_1(\hat{h}_i) \\ t_{(k+1,0)} &= z_\pm(\hat{h}_i, \hat{h}_{i-1}, V), \end{aligned}$$

We summarize the above three cases by

$$\hat{h}_{i+1}(x) = z_\pm(\hat{h}_i(x), \hat{h}_{i-1}(x), V) S_{p_i+1}(\hat{h}_i(x)) - \hat{h}_{i-1}(x) S_{p_i}(\hat{h}_i(x)), \tag{2.10}$$

where p_i take values as follows,

$$p_i = \begin{cases} a_{k+1}, & \text{if } \hat{B}_i \text{ is of } (k, III)\text{-type and } \hat{B}_{i+1} \text{ is of } (k+1, I)\text{-type}, \\ a_{k+1} - 1, & \text{if } \hat{B}_i \text{ is of } (k, III)\text{-type and } \hat{B}_{i+1} \text{ is of } (k+1, III)\text{-type}, \\ a_{k+1} + 1, & \text{if } \hat{B}_i \text{ is of } (k, II)\text{-type and } \hat{B}_{i+1} \text{ is of } (k+1, I)\text{-type}, \\ a_{k+1}, & \text{if } \hat{B}_i \text{ is of } (k, II)\text{-type and } \hat{B}_{i+1} \text{ is of } (k+1, III)\text{-type}, \\ 1, & \text{if } \hat{B}_i \text{ is of } (k, I)\text{-type or an added rung at order } k. \end{cases} \tag{2.11}$$

Definition 2. *For $p \geq 1$, $1 \leq l \leq p$, set*

$$I_{p,l} = \left\{ 2\cos\frac{l+c}{p+1}\pi \;:\; |c| \leq \frac{1}{10},\ |S_{p+1}(2\cos\frac{l+c}{p+1}\pi)| \leq \frac{1}{4} \right\}.$$

By the definition, for any $1 \leq l \leq p$, we have $S_{p+1}(2\cos\frac{l\pi}{p+1}) = 0$; $|S_{p+1}(2\cos\frac{l+c}{p+1}\pi)| \leq \frac{1}{4}$ implies $|c| \leq \frac{1}{10}$; $\{I_{p,l}\}_{l=1}^p$ are p disjoint intervals in $[-2, 2]$.

The following property comes from essentially [12] and [9], for the completeness, we give a proof here.

Proposition A 2. *Assume $V > 20$. Let $(\hat{B}_i)_{i=0}^m$ be a modified ladder, $(\hat{h}_i)_{i=0}^m$ the corresponding generating polynomials, and $(p_i)_{i=1}^{m-1}$ be given as in (2.11). Then for any $0 < i < m$, there exist a unique $l (1 \leq l \leq p)$ such that*

$$\hat{h}_i(\hat{B}_{i+1}) \subset I_{p_i,l}.$$

Proof. For convenience, we denote $z_\pm(\hat{h}_i(x), \hat{h}_{i-1}(x), V)$ by $z_\pm(x)$.

Note that $S_p^2 - 1 = S_{p-1} S_{p+1}$. For $\delta = \pm 1$, by (2.10) and a direct computation,

$$(S_p(\hat{h}_i) + \delta)(\hat{h}_{i+1} + \delta \hat{h}_{i-1}) = S_{p+1}(\hat{h}_i) \left(\left(z_\pm(x) S_p(\hat{h}_i) - \hat{h}_{i-1} S_{p-1}(\hat{h}_i) \right) + \delta z_\pm(x) \right). \tag{2.12}$$

Notice first for any $x \in \hat{B}_{i+1}$, we have $|z_\pm(x)| \geq V - 2$. On the other hand, it can be verified that

$$\Lambda(\hat{h}_{i+1}, \hat{h}_i, z_\pm(x) S_p(\hat{h}_i) - \hat{h}_{i-1} S_{p-1}(\hat{h}_i)) = V^2,$$

so for any $x \in \hat{B}_{i+1}$ we also have

$$|z_\pm(x)S_p(\hat{h}_i) - \hat{h}_{i-1}S_{p-1}(\hat{h}_i)| \geq V-2.$$

Choosing suitably $\delta = 1$ or -1 so that for any $x \in \hat{B}_{i+1}$,

$$\left|\Big(z_\pm S_p(\hat{h}_i) - \hat{h}_{i-1}S_{p-1}(\hat{h}_i)\Big) + \delta z_\pm\right| \geq 2(V-2).$$

Since $\hat{h}_{i+1}$ and $\hat{h}_{i-1}$ are monotone on $\hat{B}_{i+1}$, and

$$\hat{h}_{i+1}(\hat{B}_{i+1}) = [-2,2], \quad \hat{h}_{i-1}(\hat{B}_{i+1}) \subset [-2,2],$$

there exists a unique point $x_0 \in \hat{B}_{i+1}$ such that

$$\hat{h}_{i+1}(x_0) + \delta\hat{h}_{i-1}(x_0) = 0.$$

By the above discussions and (2.12), $\hat{h}_i(x_0)$ must be a zero point of $S_{p+1}|_{[-2,2]}$, then there is a unique $1 \leq l \leq p$ such that

$$\hat{h}_i(x_0) = 2\cos\frac{l\pi}{p+1}.$$

For any $y_j \in \hat{B}_i$ with $\hat{h}_i(y_j) = 2\cos\frac{j\pi}{p}$, $j = 1, \cdots, p-1$, by (2.10) and a simple computation, we get

$$|\hat{h}_{i+1}(y_j)| = |z_\pm(\hat{h}_i(y_j), \hat{h}_{i-1}(y_j), V)| \geq V-2,$$

which yields $\hat{h}_i(\hat{B}_{i+1}) \subset 2\cos[\frac{l-1}{p}\pi, \frac{l}{p}\pi]$. Hence, for any $x \in \hat{B}_{i+1}$, there exists a unique c with $|c| < 1$ such that $\hat{h}_i(x) = 2\cos\frac{l+c}{p+1}\pi$.

For any $x \in \hat{B}_{i+1}$, by $|S_p(2\cos\theta)| \leq |S_{p+1}(2\cos\theta)| + 1$,

$$\begin{aligned}
2 &\geq |\hat{h}_{i+1}(x)| \\
&\geq |z_\pm(x)S_{p+1}(\hat{h}_i(x))| - |\hat{h}_{i-1}(x)S_p(\hat{h}_i(x))| \\
&\geq (V-2)|S_{p+1}(\hat{h}_i(x))| - (|S_{p+1}(\hat{h}_i(x))| + 1)|\hat{h}_{i-1}(x)| \\
&= (V-4)|S_{p+1}(\hat{h}_i(x))| - 2,
\end{aligned}$$

which follows that $|S_{p+1}(\hat{h}_i(x))| \leq \frac{4}{V-4}$. Hence if $V > 20$, we have $|S_{p+1}(\hat{h}_i(x))| \leq \frac{1}{4}$. A direct computation gives also

$$S_{p+1}(\hat{h}_i(x)) = S_{p+1}(2\cos\frac{(l+c)\pi}{p+1}) = \frac{\sin(l+c)\pi}{\sin\frac{l+c}{p+1}\pi} = \frac{(-1)^l\sin c\pi}{\sin\frac{l+c}{p+1}\pi}.$$

we get finally $|c| \leq \frac{1}{10}$. □

Definition 3. *Assume $V > 20$. Let $(\hat{B}_i)_{i=0}^m$ be a modified ladder. Let $(p_i)_{i=1}^{m-1}$ and $(l_i)_{i=1}^{m-1}$ be given as in* (2.11) *and Proposition A2 respectively, which will be called the* type sequence *and the* index sequence *respectively with respect to the modified ladder.*

Note that if $\alpha = [0; a_1, a_2, \cdots]$ with $(a_n)_{n\geq 1}$ bounded by a constant M, then the type sequence $(p_i)_{i=1}^{m-1}$ is bounded by $M+1$.

Lemma A3. ([12, 9]) *For $V > 4$, $k \geq 0$,*

(1) *A (k, I)-type band contains a unique band of $\sigma_{(k+2,0)}$ which is a $(k+1, \mathrm{II})$-type band;*

(2) *Let $B_{(k+1,0)}$ be a (k, II)-type band.*

$B_{(k+1,0)}$ contains $a_{k+1}+1$ bands of $\sigma_{(k+1,1)}$ which are of $(k+1, \mathrm{I})$-type, note that the fact

$$t_{(k+1,0)}(B^{(i)}_{(k+1,1)}) \subset I_{a_{k+1}+1,i} \ , \quad i = 1, \cdots, a_{k+1}+1,$$

we can index these bands as $\{B^{(i)}_{(k+1,1)}\}_{i=1}^{a_{k+1}+1}$.

$B_{(k+1,0)}$ contains a_{k+1} bands of $\sigma_{(k+2,0)}$, which are of $(k+1, \mathrm{III})$-type, and we can index them as $\{B^{(i)}_{(k+2,0)}\}_{i=1}^{a_{k+1}}$ by the fact

$$t_{(k+1,0)}(B^{(i)}_{(k+2,0)}) \subset I_{a_{k+1},i} \ , \quad i = 1, \cdots, a_{k+1}.$$

(3) *Let $B_{(k+1,0)}$ be a (k, III)-type band.*

$B_{(k+1,0)}$ contains a_{k+1} bands of $\sigma_{(k+1,1)}$, which are of $(k+1, \mathrm{I})$-type, and we can index them as $\{B^{(i)}_{(k+1,1)}\}_{i=1}^{a_{k+1}}$ by the fact

$$t_{(k+1,0)}(B^{(i)}_{(k+1,1)}) \subset I_{a_{k+1},i} \ , \quad i = 1, \cdots, a_{k+1}.$$

$B_{(k+1,0)}$ contains $a_{k+1}-1$ bands of $\sigma_{(k+2,0)}$, which are of $(k+1, \mathrm{III})$-type, and we can index them as $\{B^{(i)}_{(k+2,0)}\}_{i=1}^{a_{k+1}-1}$ by the fact

$$t_{(k+1,0)}(B^{(i)}_{(k+2,0)}) \subset I_{a_{k+1}-1,i} \ , \quad i = 1, \cdots, a_{k+1}-1.$$

We summarize the estimation of Chebischev polynomials on the interval $I_{p,l}$, which has been got in the proof of Proposition 7 of [9].

Proposition A4. *Fix $p \geq 1$, $1 \leq l \leq p$. For $V > 20$, and any $t \in I_{p,l}$,*

$$\begin{array}{l} |S_{p+1}(t)| \leq \frac{1}{4}, \quad |S_p(t)| \leq \frac{5}{4}, \\ \frac{p+1}{3} \leq |S'_{p+1}(t)| \leq \frac{(p+1)^3}{4}, \quad |S'_p(t)| \leq 2|S'_{p+1}(t)|. \end{array}$$

With above discussions, we can simplify part of the statement of Proposition 7,8,9 of [9] as the following, which is got by Proposition A4 and (2.10).

Proposition A5 ([9])**.** *Assume $V > 20$. Let $(\hat{B}_i)_{i=0}^m$ be a modified ladder, $(\hat{h}_i)_{i=0}^m$ the corresponding generating polynomials and $(p_i)_{i=1}^{m-1}$ the corresponding type sequence. For any $0 < i < m$, we have,*

$$(V-8)\frac{(p_i+1)}{3} \leq \left|\frac{\hat{h}'_{i+1}(x)}{\hat{h}'_i(x)}\right| \leq (V+8)\frac{(p_i+1)^3}{4}.$$

Now we state the results of this paper which are motivated from [11].

Fix $\alpha = [0; a_1, a_2, a_3, \cdots]$ with $(a_n)_{n\geq 1}$ bounded by $M(\geq 1)$.

Theorem 1 (Bounded variation)**.** *Let $V > 20$, B_n be a spectral generating band of order n with generating polynomial h_n. There exists a constant $\xi \geq 1$ independent of n and B_n such that, for any $x_1, x_2 \in B_n$,*

$$\xi^{-1} \leq \left|\frac{h'_n(x_1)}{h'_n(x_2)}\right| \leq \xi.$$

Corollary 2 (Bounded distortion)**.** *Let $V > 20$, B_n be a spectral generating band of order n with generating polynomial h_n. Then for any $x \in B_n$,*

$$\xi^{-1} \leq |h_n'(x)| \cdot |B_n| \leq \xi.$$

Theorem 3 (Bounded covariation)**.** *Suppose V is sufficiently large. Given $n > k \geq 1$, let*

$$\begin{array}{c} B_n \subset \cdots \subset B_{k+1} \subset B_k \\ \tilde{B}_n \subset \cdots \subset \tilde{B}_{k+1} \subset \tilde{B}_k, \end{array}$$

be two sequence of spectral generating bands. For any $k+1 \leq i \leq n$, B_i and $\tilde{B}_i$ are of same type and have the same index. B_k and $\tilde{B}_k$ are of same type. So, there exists $\eta > 1$ such that

$$\eta^{-1}\frac{|\tilde{B}_n|}{|\tilde{B}_k|} \leq \frac{|B_n|}{|B_k|} \leq \eta\frac{|\tilde{B}_n|}{|\tilde{B}_k|}.$$

Theorem 4 (Existence of Gibbs-like measures)**.** *Suppose V is sufficiently large. Given $\beta > 0$, there exist $\zeta > 0$ and a probability measure μ_β supported by $\sigma(H_{\alpha,V})$ such that for any $k \geq 1$ and $\tilde{B} \in \mathscr{G}_k$,*

$$\zeta^{-1}\frac{|\tilde{B}|^\beta}{\sum\limits_{B\in\mathscr{G}_k}|B|^\beta} \leq \mu_\beta(\tilde{B}) \leq \zeta\frac{|\tilde{B}|^\beta}{\sum\limits_{B\in\mathscr{G}_k}|B|^\beta}.$$

Let s_n be the n-th pre-dimension of $\sigma(H_{\alpha,V})$, i.e.,

$$\sum_{B\in\mathscr{G}_n}|B|^{s_n} = 1,$$

and

$$s_* = \liminf_{n\to\infty} s_n, \quad s^* = \limsup_{n\to\infty} s_n.$$

Theorem 5. *For sufficiently large V, $\dim_H \sigma(H_{\alpha,V}) = s_*$.*

Remark: Liu, Peyrière and Wen proved in [7] that $\overline{\dim}_B\,\sigma(H_{\alpha,V}) = s^*$, but there was a small error there (Page 670 in [7]), we can correct it as follows:

letting $B_n \subset B_{n-1}$ be two spectral generating bands of order n and $n-1$, taking notation of [7], denoting B_n as J, B_{n-1} as J^{-1}, the inequality

$$q_{ij}(n) \leq \frac{|J|}{|J^{-1}|} \leq p_{ij}(n)$$

should be replaced by the inequality(see [9], in proof of Proposition 5, (4.36)-(4.40))

$$q_{ij}(n) \leq \frac{|h_{n-1}'(x)|}{|h_n'(x)|} \leq p_{ij}(n), \quad \forall x \in J,$$

where h_n and h_{n-1} are the corresponding generating polynomial of B_n and B_{n-1}.

From above inequality, applying Corollary 2 we get

$$\frac{q_{ij}(n)}{\xi^2} \leq \frac{|J|}{|J^{-1}|} \leq \min\{\xi^2 p_{ij}(n), 1\},$$

which yields the lower bound of contractive ratio is strictly larger than 0. Then as in the proof of [7], we still have

Theorem A6. *For $V>20$, $\overline{\dim}_B\,\sigma(H_{\alpha,V})=s^*$.*

3. Bounded variation, Bounded covariation, Gibbs measure

From Proposition A5, we get immediately the following corollary.

Corollary 6. *Assume $V>20$. Let $(\hat{B}_i)_{i=0}^m$ be a modified ladder and $(\hat{h}_i)_{i=0}^m$ the correspondent generating polynomials. Then for any $x,y\in\hat{B}_m$,*

$$|\hat{h}_i(x)-\hat{h}_i(y)|\le 3^{-(m-i)}|\hat{h}_m(x)-\hat{h}_m(y)|\le 4\cdot 3^{-(m-i)}.$$

Proof. For any $0\le i\le m$, since $\hat{h}_i$ is monotone on $\hat{B}_i$

$$\begin{aligned}|\hat{h}_i(x)-\hat{h}_i(y)| &= \left|\int_x^y \hat{h}_i'(t)dt\right|\\ &= \left|\int_x^y \frac{\hat{h}_i'(t)}{\hat{h}_{i+1}'(t)}\hat{h}_{i+1}'(t)dt\right|\\ &\le 3^{-1}\left|\int_x^y \hat{h}_{i+1}'(t)dt\right|\\ &= 3^{-1}|\hat{h}_{i+1}(x)-\hat{h}_{i+1}(y)|,\end{aligned}$$

where the inequality is due to Proposition A5. □

Proposition 7. *Assume $V>20$. Let $(\hat{B}_i)_{i=0}^m$ and $(\hat{\tilde{B}}_i)_{i=0}^m$ be two modified ladders. Suppose that they have the same sequence of generating polynomials $(\hat{h}_i)_{i=0}^m$ and the same type sequence $(p_i)_{i=1}^{m-1}$.*

Suppose $x_1\in\hat{B}_m$, $x_2\in\hat{\tilde{B}}_m$, $0<i<m$.

In the case of $\hat{B}_i$ is not a band of order 0, then

$$\begin{aligned}\left|\frac{\hat{h}_{i+1}'(x_1)}{h_i'(x_1)}-\frac{\hat{h}_{i+1}'(x_2)}{\hat{h}_i'(x_2)}\right| &\le C(|\hat{h}_i(x_1)-\hat{h}_i(x_2)|+|\hat{h}_{i-1}(x_1)-\hat{h}_{i-1}(x_2)|)+\\ &\qquad \frac{1}{3}\left|\frac{\hat{h}_i'(x_1)}{\hat{h}_{i-1}'(x_1)}-\frac{\hat{h}_i'(x_2)}{\hat{h}_{i-1}'(x_2)}\right|,\end{aligned}\tag{3.1}$$

where C is a constant depending on V and p_i.

In the case of $\hat{B}_i$ is a band of order 0, then

$$\left|\frac{\hat{h}_{i+1}'(x_1)}{h_i'(x_1)}-\frac{\hat{h}_{i+1}'(x_2)}{\hat{h}_i'(x_2)}\right| \le C|\hat{h}_i(x_1)-\hat{h}_i(x_2)|.\tag{3.2}$$

Proof. Take any $0<i<m$, for convenience, we denote $z_\pm(\hat{h}_i(x),\hat{h}_{i-1}(x),V)$ as $z_\pm(x)$.

Suppose first $\hat{B}_i$ is not a band of order 0.

By taking derivative on both side of (2.10), we get

$$\begin{aligned}\frac{\hat{h}_{i+1}'(x)}{\hat{h}_i'(x)} &= S_{p_i+1}'(\hat{h}_i(x))z_\pm(x)-S_{p_i}'(\hat{h}_i(x))\hat{h}_{i-1}(x)+\\ &\qquad S_{p_i+1}(\hat{h}_i(x))\frac{z_\pm'(x)}{\hat{h}_i'(x)}-S_{p_i}(\hat{h}_i(x))\frac{\hat{h}_{i-1}'(x)}{\hat{h}_i'(x)}.\end{aligned}\tag{3.3}$$

Observing that $S_{p+1}(x)$ is a polynomial of degree p, $S''_{p+1}|_{[-2,2]}$ is also bounded by some constant depends on p. So, there exists a constant $c_1 > 0$ depending only on p_i such that

$$\begin{aligned}
&\left|S_{p_i+1}(\hat{h}_i(x_1)) - S_{p_i+1}(\hat{h}_i(x_2))\right| \leq c_1|\hat{h}_i(x_1) - \hat{h}_i(x_2)| \\
&\left|S'_{p_i+1}(\hat{h}_i(x_1)) - S'_{p_i+1}(\hat{h}_i(x_2))\right| \leq c_1|\hat{h}_i(x_1) - \hat{h}_i(x_2)| \\
&\left|S_{p_i}(\hat{h}_i(x_1)) - S_{p_i}(\hat{h}_i(x_2))\right| \leq c_1|\hat{h}_i(x_1) - \hat{h}_i(x_2)| \\
&\left|S'_{p_i}(\hat{h}_i(x_1)) - S'_{p_i}(\hat{h}_i(x_2))\right| \leq c_1|\hat{h}_i(x_1) - \hat{h}_i(x_2)| \\
&|z_\pm(x_1) - z_\pm(x_2)| \leq |\hat{h}_i(x_1) - \hat{h}_i(x_2)| + |\hat{h}_{i-1}(x_1) - \hat{h}_{i-1}(x_2)|
\end{aligned} \tag{3.4}$$

where the last inequality is due to the fact (2.9), and

$$\frac{z'_\pm(x)}{\hat{h}'_i(x)} = z_1(\hat{h}_i(x), \hat{h}_{i-1}(x), V) + z_2(\hat{h}_i(x), \hat{h}_{i-1}(x), V)\frac{\hat{h}'_{i-1}(x)}{\hat{h}'_i(x)}. \tag{3.5}$$

By (2.9), we have

$$\begin{aligned}
&|z_1(\hat{h}_i(x_1), \hat{h}_{i-1}(x_1), V) - z_1(\hat{h}_i(x_2), \hat{h}_{i-1}(x_2), V)| \\
&\quad \leq |\hat{h}_i(x_1) - \hat{h}_i(x_2)| + |\hat{h}_{i-1}(x_1) - \hat{h}_{i-1}(x_2)| \\
&|z_2(\hat{h}_i(x_1), \hat{h}_{i-1}(x_1), V) - z_2(\hat{h}_i(x_2), \hat{h}_{i-1}(x_2), V)| \\
&\quad \leq |\hat{h}_i(x_1) - \hat{h}_i(x_2)| + |\hat{h}_{i-1}(x_1) - \hat{h}_{i-1}(x_2)|.
\end{aligned} \tag{3.6}$$

By a direct computation,

$$\begin{aligned}
\left|\frac{\hat{h}'_{i-1}(x_1)}{\hat{h}'_i(x_1)} - \frac{\hat{h}'_{i-1}(x_2)}{\hat{h}'_i(x_2)}\right| &= \left|\frac{\hat{h}'_{i-1}(x_1)}{\hat{h}'_i(x_1)}\frac{\hat{h}'_{i-1}(x_2)}{\hat{h}'_i(x_2)}\right|\left|\frac{\hat{h}'_i(x_1)}{\hat{h}'_{i-1}(x_1)} - \frac{\hat{h}'_i(x_2)}{\hat{h}'_{i-1}(x_2)}\right| \\
&\leq \frac{1}{9}\left|\frac{\hat{h}'_i(x_1)}{\hat{h}'_{i-1}(x_1)} - \frac{\hat{h}'_i(x_2)}{\hat{h}'_{i-1}(x_2)}\right|.
\end{aligned} \tag{3.7}$$

The inequalities (3.3)-(3.7) imply that the inequality (3.1) holds.

Suppose $\hat{B}_i$ is a band of order 0. Note that $\hat{h}_{i-1} = t_{(0,0)} \equiv 2$ is a constant, then an analogous argument to (3.3)-(3.6) implies that the inequality (3.2) holds. $\square$

Proof of Theorem 1. It is a corollary of Corollary 6 and Proposition 7. In fact, let

$$B_n \subset B_{n-1} \subset \cdots \subset B_0 \subset B_{-1}$$

be a sequence of spectral generating bands(the orders are from n to -1), which form an initial ladder. Let $(\hat{B}_i)_{i=-1}^m$ be the corresponding modified ladder, $(\hat{h}_i)_{i=-1}^m$ the corresponding generating polynomials. By Corollary 6, for any $0 \leq i < n$,

$$|\hat{h}_i(x_1) - \hat{h}_i(x_2)| \leq 4 \cdot 3^{-(m-i)}.$$

Note that $\hat{B}_0 = B_0$, $\hat{B}_{-1} = B_{-1}$, we have $\hat{h}'_0 \equiv 1$, thus

$$|\log|\hat{h}'_m(x_1)| - \log|\hat{h}'_m(x_2)|| \leq \sum_{i=0}^{m-1}\left|\log\left|\frac{\hat{h}'_{i+1}(x_1)}{\hat{h}'_i(x_1)}\right| - \log\left|\frac{\hat{h}'_{i+1}(x_2)}{\hat{h}'_i(x_2)}\right|\right| \tag{3.8}$$

and

$$\left|\log\left|\frac{\hat{h}'_{i+1}(x_1)}{\hat{h}'_i(x_1)}\right| - \log\left|\frac{\hat{h}'_{i+1}(x_2)}{\hat{h}'_i(x_2)}\right|\right| \le \left|\frac{\hat{h}'_i(x_2)}{\hat{h}'_{i+1}(x_2)}\right|\left|\frac{\hat{h}'_{i+1}(x_1)}{\hat{h}'_i(x_1)} - \frac{\hat{h}'_{i+1}(x_2)}{\hat{h}'_i(x_2)}\right|. \tag{3.9}$$

$\hat{B}_0$ is a order 0 band, so by (3.2)

$$\left|\frac{\hat{h}'_1(x_1)}{\hat{h}'_0(x_1)} - \frac{\hat{h}'_1(x_2)}{\hat{h}'_0(x_2)}\right| \le C|\hat{h}_0(x_1) - \hat{h}_0(x_2)| \le 4C \cdot 3^{-m}.$$

Combining with (3.1) and induction, we have for $0 \le i < m$,

$$\left|\frac{\hat{h}'_{i+1}(x_1)}{\hat{h}'_i(x_1)} - \frac{\hat{h}'_{i+1}(x_2)}{\hat{h}'_i(x_2)}\right| \le 8C \cdot 3^{-(m-i)},$$

together with (3.8) and (3.9), we finish the proof of the theorem. □

Proposition 8. *Suppose that $(\hat{B}_i)_{i=0}^m$ and $(\hat{\hat{B}}_i)_{i=0}^m$ are two modified ladders having the same sequence of generating polynomials $(\hat{h}_i)_{i=0}^m$, the same type sequence $(p_i)_{i=1}^{m-1}$(bounded by $M+1$), and the same index sequence $(l_i)_{i=1}^{m-1}$. Then for sufficiently large V, we have*

(i) *There exists $c < \frac{1}{4}$ such that for any $0 < i < m$ and any $x_1 \in \hat{B}_{i+1}$, $x_2 \in \hat{\hat{B}}_{i+1}$,*

$$|\hat{h}_i(x_1) - \hat{h}_i(x_2)| \le c(|\hat{h}_{i+1}(x_1) - \hat{h}_{i+1}(x_2)| + |\hat{h}_{i-1}(x_1) - \hat{h}_{i-1}(x_2)|). \tag{3.10}$$

(ii) *Letting $\lambda = \frac{1+\sqrt{1-4c^2}}{2c} (> 1)$, for any $x_1 \in \hat{B}_m$, there exists $x_2 \in \hat{\hat{B}}_m$ such that, for any $0 \le i \le m$,*

$$|\hat{h}_i(x_1) - \hat{h}_i(x_2)| \le \frac{4\lambda^2}{\lambda^2 - 1}\lambda^{-i}. \tag{3.11}$$

(iii) *There exists $\eta > 1$ such that*

$$\eta^{-1}\frac{|\hat{\hat{B}}_m|}{|\hat{\hat{B}}_0|} \le \frac{|\hat{B}_m|}{|\hat{B}_0|} \le \eta\frac{|\hat{\hat{B}}_m|}{|\hat{\hat{B}}_0|}.$$

Proof. (i) Take any $0 < i < m$ and any $x_1 \in \hat{B}_{i+1}$, $x_2 \in \hat{\hat{B}}_{i+1}$. For convenience, we denote $z_\pm(\hat{h}_i(x), \hat{h}_{i-1}(x), V)$ by $z_\pm(x)$.

By the definitions of p_i and l_i, $\hat{h}_i(x_1)$, $\hat{h}_i(x_2) \in I_{p_i,l_i}$, then by Proposition A4,

$$\left|S_{p_i+1}(\hat{h}_i(x_1)) - S_{p_i+1}(\hat{h}_i(x_2))\right| \ge \frac{p_i+1}{3}|\hat{h}_i(x_1) - \hat{h}_i(x_2)|. \tag{3.12}$$

By Proposition A4 again,

$$\left|S_p(\hat{h}_i(x_1)) - S_p(\hat{h}_i(x_2))\right| \le \frac{(p+1)^3}{4}|\hat{h}_i(x_1) - \hat{h}_i(x_2)|.$$

By (2.9) as in (3.4),

$$|z_\pm(x_1)) - z_\pm(x_2))| \le |\hat{h}_i(x_1) - \hat{h}_i(x_2)| + |\hat{h}_{i-1}(x_1) - \hat{h}_{i-1}(x_2)|.$$

So by the above three inequalities and (2.10), we get

$$\begin{aligned}|\hat{h}_{i+1}(x_1)-\hat{h}_{i+1}(x_2)| \geq\ & \tfrac{(V-2)(p+1)}{3}|\hat{h}_i(x_1)-\hat{h}_i(x_2)|-\tfrac{1}{4}|\hat{h}_i(x_1)-\hat{h}_i(x_2)|\\ &-\tfrac{6}{4}|\hat{h}_{i-1}(x_1)-\hat{h}_{i-1}(x_2)|-(p+1)^3|\hat{h}_i(x_1)-\hat{h}_i(x_2)|,\end{aligned}$$

which concludes the inequality (3.10).

(ii) Since

$$\hat{h}_m(\hat{B}_m)=[-2,2],\ \hat{h}_m(\hat{\hat{B}}_m)=[-2,2],$$

for any $x_1\in\hat{B}_m$, there exists $x_2\in\hat{\hat{B}}_m$ such that

$$\hat{h}_m(x_1)=\hat{h}_m(x_2).$$

Since for any $0\le i<m$,

$$\hat{h}_i(\hat{B}_m)\subset[-2,2],\ \hat{h}_i(\hat{\hat{B}}_m)\subset[-2,2],$$

we get for any $0\le i<m$,

$$|\hat{h}_i(x_1)-\hat{h}_i(x_2)|\le 4.$$

Let

$$f_i=|\hat{h}_i(x_1)-\hat{h}_i(x_2)|,\quad 0\le i\le m,$$

then $f_m=0,\ f_0\le 4$. By (3.10),

$$0\le(\lambda f_{m-1}-f_m)\le\lambda^{-1}(\lambda f_{m-2}-f_{m-1})\le\cdots\le\lambda^{-m+1}(\lambda f_0-f_1)\le 4\lambda^{-m+2},$$

which implies that for any $0\le i\le m$

$$|\hat{h}_i(x_1)-\hat{h}_i(x_2)|=f_i\le\frac{4\lambda^2}{\lambda^2-1}\lambda^{-i}.$$

(iii) By (3.11), an argument similar to Theorem 1 implies there exist $\xi_1>1$ such that

$$\xi_1^{-1}\le\left|\frac{\hat{h}'_m(x_1)/\hat{h}'_0(x_1)}{\hat{h}'_m(x_2)/\hat{h}'_0(x_2)}\right|\le\xi_1. \tag{3.13}$$

By the definition of the generating polynomial, there exist $\tilde{x}\in\hat{B}_m$, $\tilde{y}\in\hat{B}_0$ such that

$$|\hat{B}_m|\,|\hat{h}'_m(\tilde{x})|=4,\ |\hat{B}_0|\,|\hat{h}'_0(\tilde{y})|=4.$$

Associating with Theorem 1, we have

$$\frac{|\hat{B}_m|}{|\hat{B}_0|}=\frac{|\hat{B}_m|\,|\hat{h}'_m(\tilde{x})|}{|\hat{B}_0|\,|\hat{h}'_0(\tilde{y})|}\left|\frac{\hat{h}_m(x_1)}{\hat{h}_m(\tilde{x})}\right|\left|\frac{\hat{h}_0(\tilde{y})}{\hat{h}_0(x_1)}\right|\left|\frac{\hat{h}_0(x_1)}{\hat{h}_m(x_1)}\right|\le\xi^2\left|\frac{\hat{h}_0(x_1)}{\hat{h}_m(x_1)}\right|.$$

By the same discussion, we have

$$\frac{|\hat{\hat{B}}_m|}{|\hat{\hat{B}}_0|}\ge\xi^{-2}\left|\frac{\hat{h}_0(x_2)}{\hat{h}_m(x_2)}\right|.$$

Then by (3.13), we have

$$\frac{|\hat{B}_m|}{|\hat{B}_0|}\le\xi^4\xi_1\frac{|\hat{\hat{B}}_m|}{|\hat{\hat{B}}_0|}.$$

The opposite direction of the inequality can be got by the same way. □

Proof of Theorem 3. Let $(\hat{B}_i)_{i=0}^m$ be the modified ladder of initial ladder $(B_i)_{i=k}^n$ and $(\hat{\tilde{B}}_i)_{i=0}^m$ be the modified ladder of the initial ladder $(\tilde{B}_i)_{i=k}^n$.

Since for $k \le i \le m$, B_i and $\tilde{B}_i$ are of the same type, $(\hat{B}_i)_{i=0}^m$ and $(\hat{\tilde{B}}_i)_{i=0}^m$ share the same sequence of generating polynomials $(\hat{h}_i)_{i=0}^m$ and the same type sequence $(p_i)_{i=1}^{m-1}$.

Since for $k < i \le m$, B_i and $\tilde{B}_i$ are of the same index, $(\hat{B}_i)_{i=0}^m$ and $(\hat{\tilde{B}}_i)_{i=0}^m$ share the same index sequence $(l_i)_{i=1}^{m-1}$.

Then Proposition 8 concludes the result of the theorem. □

Given a spectral generating band B_n of order n, there exists a unique sequence of spectral generating bands $(B_i)_{i=0}^{n-1}$ so that

$$B_n \subset B_{n-1} \subset \cdots B_1 \subset B_0,$$

we are going to define the *characteristic index* $i_0 i_1 \cdots i_n$ of B_n as follows, fix $0 \le k \le n-1$,

Case 1: B_k is a (k, II)-type band.
If B_{k+1} is $(k+1, I)$ type band with index j, then define $i_{k+1} := (I, j)$;
if B_{k+1} is $(k+1, III)$ type band with index j, then define $i_{k+1} := (III, j)$.

Case 2: B_k is a (k, III)-type band.
If B_{k+1} is $(k+1, I)$ type band with index j, then $i_{k+1} := (I, j)$;
if B_{k+1} is $(k+1, III)$ type band with index j, then $i_{k+1} := (III, j)$.

Case 3: B_k is a (k, I)-type band, then $i_{k+1} := (II)$.

Case 4: If B_0 is of $(0, I)$-type, then $i_0 := (I)$; if B_0 is of $(0, III)$-type, then $i_0 = (III)$.

We call $i_0 i_1 \cdots i_m$ an *admissible index* (of length m) if it is a characteristic index of a band B_m of order m. Denote by Ω_m the set of all admissible index of length m. For any admissible index $\omega \in \Omega_m$, there is only one associated spectral generating band, which we denoted as B_ω. For any $i_0 \cdots i_m \in \Omega_m$ and any $0 \le j \le m$, we call the symbol i_j is of type I (II or III), if the corresponding band $B_{i_0 \cdots i_j} \in \Omega_j$ is of type I (II or III respectively). Now we give some more notations:

- $\Omega_{k,m}$: all segments $i_k \cdots i_m$ of any admissible index $i_0 i_1 \cdots i_m \in \Omega_m$, $m \ge k > 0$.
- $\Omega_{k+1,m}^{i_k}$: all segments $i_{k+1} \cdots i_m$ of $i_0 i_1 \cdots i_k i_{k+1} \cdots i_m \in \Omega_m$, $i_k \in \Omega_{k,k}$.
  Since it depends only on type of i_k, for the convenience, we denoted it sometimes by $\Omega_{k+1,m}^{I}$,$\Omega_{k+1,m}^{II}$, or $\Omega_{k+1,m}^{III}$.
- $\Omega_m^{(k_1,j_1)(k_2,j_2)\cdots(k_l,j_l)}$: all $i_0 \cdots i_m \in \Omega_m$ satisfying $i_{k_s} = j_s$ with $j_s \in \Omega_{k_s,k_s}$ for $1 \le s \le l$.

For any $0 < \beta < 1$ and $m > 0$, we define a probability $\mu_{\beta,m}$ on $\mathbb{R}$ such that for any $\omega_0 \in \Omega_m$,

$$\mu_{\beta,m}(B_{\omega_0}) = \frac{|B_{\omega_0}|^\beta}{\sum\limits_{\omega \in \Omega_m} |B_\omega|^\beta},$$

where $\mu_{\beta,m}$ is uniformly distributed on each band B_ω. For the convenience, for any $m > 0$, denote

$$b_m := \sum_{\omega \in \Omega_m} |B_\omega|^\beta.$$

For any $k \geq 1$, any $\omega = i_0 i_1 \cdots i_k \in \Omega_k$ and any $m > k$, we have

$$\mu_{\beta,m}(B_\omega) = \sum_{\sigma \in \Omega^{i_k}_{k+1,m}} \mu_{\beta,m}(B_{\omega*\sigma}),$$

where $\omega * \sigma$ is the concatenation of ω and σ.

In the following, we suppose that V is large enough so that Bounded covariation holds.

Proposition 9. *Let $\mu_{\beta,m}$ be defined as above. Then there exists $c \geq 1$ such that*

(i) *for any $k > 0$ and $\omega \in \Omega_k$,*

$$c^{-1}\mu_{\beta,k+3}(B_\omega) \leq \mu_{\beta,k}(B_\omega) \leq c\mu_{\beta,k+3}(B_\omega); \tag{3.14}$$

(ii) *for any $k > 0$, $m > k+3$, $\omega = i_0 \cdots i_k \in \Omega_k$, $\sigma \in \Omega^{i_k}_{k+1,k+3}$,*

$$\mu_{\beta,m}(B_\omega) \leq c\mu_{\beta,m}(B_{\omega*\sigma}). \tag{3.15}$$

Proof. (i) Take any $\omega_0 = i_0 \cdots i_k \in \Omega_k$. For any $\sigma \in \Omega^{i_k}_{k+1,k+3}$, $x \in B_{\omega_0*\sigma}$, by Corollary 2 and Proposition A5,

$$1 \leq \frac{|B_{\omega_0}|}{|B_{\omega_0*\sigma}|} \leq \xi^2 \frac{|h'_{k+3}(x)|}{|h'_k(x)|} \leq \xi^2((M+2)^3(V+8))^{2M+1},$$

where $h_k(x)$ is the generating polynomial of B_{ω_0} and $h_{k+3}(x)$ is the generating polynomial of $B_{\omega_0*\sigma}$, and the length of modified ladder from B_{ω_0} to $B_{\omega_0*\sigma}$ is at most $2M+1$ and at least 1. So for $c_1 = \xi^2((M+2)^3(V+5))^{2M+1}$,

$$1 \leq \frac{|B_{\omega_0}|}{|B_{\omega_0*\sigma}|} \leq c_1. \tag{3.16}$$

Since B_{ω_0} contains at most $(2M+1)^3$ bands of order $k+3$,

$$(2M+1)^{-3}b_{k+3} \leq b_k \leq c_1 b_{k+3}.$$

Hence, for any $\omega_0 \in \Omega_k$,

$$c_1^{-1}\mu_{\beta,k+3}(B_{\omega_0}) \leq \mu_{\beta,k}(B_{\omega_0}) \leq c_1(2M+1)^3\mu_{\beta,k+3}(B_{\omega_0}),$$

which yields the inequality (3.14).

(ii) For any $\omega_0 = i_0 \cdots i_k \in \Omega_k$ and any $m > k+3$,

$$\mu_{\beta,m}(B_{\omega_0}) = \sum_{\sigma \in \Omega^{i_k}_{k+1,k+3}} \mu_{\beta,m}(B_{\omega_0*\sigma}). \tag{3.17}$$

We will show there exists $c_2 > 1$ such that, for any $m > k+6$, $\sigma_1, \sigma_2 \in \Omega^{i_k}_{k+1,k+3}$,

$$\mu_{\beta,m}(B_{\omega_0*\sigma_1}) \leq c_2\mu_{\beta,m}(B_{\omega_0*\sigma_2}). \tag{3.18}$$

together with (3.14), (3.17) and $\sharp\Omega^{i_k}_{k+1,k+3} \leq (2M+1)^3$, we will get the inequality (3.15).

Fix $\sigma_1, \sigma_2 \in \Omega^{i_k}_{k+1,k+3}$. Let i be the last symbol of σ_1, j be the last symbol of σ_2. Divide $\Omega^i_{k+4,k+6}$ into three sets D_1, D_2, D_3 according to the last symbol being type I, type II,

or type III. Divide also the set $\Omega^{j}_{k+4,k+6}$ into three sets $\tilde{D}_1, \tilde{D}_2, \tilde{D}_3$ according to the last symbol being type I, type II, or type III. So

$$\mu_{\beta,m}(B_{\omega_0*\sigma_1}) = \sum_{\tau\in D_1} \mu_{\beta,m}(B_{\omega_0*\sigma_1*\tau}) + \sum_{\tau\in D_2} \mu_{\beta,m}(B_{\omega_0*\sigma_1*\tau}) + \sum_{\tau\in D_3} \mu_{\beta,m}(B_{\omega_0*\sigma_1*\tau}),$$
$$\mu_{\beta,m}(B_{\omega_0*\sigma_2}) = \sum_{\tau\in \tilde{D}_1} \mu_{\beta,m}(B_{\omega_0*\sigma_2*\tau}) + \sum_{\tau\in \tilde{D}_2} \mu_{\beta,m}(B_{\omega_0*\sigma_2*\tau}) + \sum_{\tau\in \tilde{D}_3} \mu_{\beta,m}(B_{\omega_0*\sigma_2*\tau}),$$

Fix any $\tau_1 \in D_1$, $\tau_2 \in \tilde{D}_1$.

$$\mu_{\beta,m}(B_{\omega_0*\sigma_1*\tau_1}) = \textstyle\sum_{\tau\in\Omega^I_{k+7,m}} \mu_{\beta,m}(B_{\omega_0*\sigma_1*\tau_1*\tau}),$$
$$\mu_{\beta,m}(B_{\omega_0*\sigma_2*\tau_2}) = \textstyle\sum_{\tau\in\Omega^I_{k+7,m}} \mu_{\beta,m}(B_{\omega_0*\sigma_2*\tau_2*\tau}).$$

By Bounded Covariant, for any $\tau \in \Omega^I_{k+7,m}$,

$$\frac{|B_{\omega_0*\sigma_1*\tau_1*\tau}|}{|B_{\omega_0*\sigma_2*\tau_2*\tau}|} \le \eta \frac{|B_{\omega_0*\sigma_1*\tau_1}|}{|B_{\omega_0*\sigma_2*\tau_2}|}$$

By (3.16), for $s = 1, 2$,

$$1 \le \frac{|B_{\omega_0}|}{|B_{\omega_0*\sigma_s*\tau_s}|} \le c_1^2.$$

So, for any $\tau \in \Omega^I_{k+7,m}$,

$$\frac{|B_{\omega_0*\sigma_1*\tau_1*\tau}|}{|B_{\omega_0*\sigma_2*\tau_2*\tau}|} \le \eta c_1^2,$$

which implies

$$\mu_{\beta,m}(B_{\omega_0*\sigma_1*\tau_1}) \le \eta c_1^2 \mu_{\beta,m}(B_{\omega_0*\sigma_2*\tau_2}).$$

The case τ_1 being of D_2,τ_2 being of $\tilde{D}_2$ respectively(and τ_1 being of D_3,τ_2 being of $\tilde{D}_3$ respectively) can be discussed by the same way. Considering that, for $i = 1, 2, 3$,

$$1 \le \sharp D_i \le (2M+1)^3, \quad 1 \le \sharp \tilde{D}_i \le (2M+1)^3,$$

we have

$$\sum_{\tau\in D_i} \mu_{\beta,m}(B_{\omega_0*\sigma_1*\tau}) \le (2M+1)^3 \eta c_1^2 \sum_{\tau\in\tilde{D}_i} \mu_{\beta,m}(B_{\omega_0*\sigma_2*\tau}).$$

This implies that the inequality (3.18) holds. □

Proof of Theorem 4. We only prove the second inequality. Let V be large enough so that Bounded covariation holds.

For any $k \ge 1$, $\omega_0 = i_0 \cdots i_k \in \Omega_k$, $m > k+3$, $\sigma = i_{k+1} i_{k+2} i_{k+3} \in \Omega^{i_k}_{k+1,k+3}$,

$$\mu_{\beta,m}(\omega_0 * \sigma) = b_m^{-1} \sum_{\sigma_1\in\Omega^{i_{k+3}}_{k+4,m}} |B_{\omega_0*\sigma*\sigma_1}|^\beta.$$

So by (3.15)

$$\mu_{\beta,m}(\omega_0) b_m \le c\mu_{\beta,m}(\omega_0*\sigma) b_m = c|B_{\omega_0*\sigma}|^\beta \sum_{\sigma_1\in\Omega^{i_{k+3}}_{k+4,m}} \frac{|B_{\omega_0*\sigma*\sigma_1}|^\beta}{|B_{\omega_0*\sigma}|^\beta}.$$

For any $\omega_1 \in \Omega_{k+3}^{(k+3,i_{k+3})}$, by bounded covariation,

$$\mu_{\beta,m}(\omega_0)b_m \le c\eta^\beta |B_{\omega_0}|^\beta \sum_{\sigma_1\in\Omega_{k+4,m}^{i_{k+3}}} \frac{|B_{\omega_1*\sigma_1}|^\beta}{|B_{\omega_1}|^\beta},$$

hence,

$$\mu_{\beta,m}(\omega_0)b_m|B_{\omega_1}|^\beta \le c\eta^\beta |B_{\omega_0}|^\beta \sum_{\sigma_1\in\Omega_{k+4,m}^{i_{k+3}}} |B_{\omega_1*\sigma_1}|^\beta.$$

Take sum on both side for any $\omega_1 \in \Omega_{k+3}^{(k+3,i_{k+3})}$,

$$\mu_{\beta,m}(\omega_0)b_m \sum_{\omega_1\in\Omega_{k+3}^{(k+3,i_{k+3})}} |B_{\omega_1}|^\beta \le c\eta^\beta |B_{\omega_0}|^\beta \sum_{\omega\in\Omega_m^{(k+3,i_{k+3})}} |B_\omega|^\beta.$$

Take sum on both sides for all $i_{k+3} \in \Omega_{k+3,k+3}$,

$$\mu_{\beta,m}(\omega_0)b_m b_{k+3} \le c\eta^\beta |B_{\omega_0}|^\beta b_m.$$

By (3.14),

$$\mu_{\beta,m}(\omega_0) \le c^2\eta^\beta b_k^{-1}|B_{\omega_0}|^\beta = c^2\eta^\beta \mu_{\beta,k}(\omega_0).$$

Let μ_β be a weak limit of $(\mu_{\beta,m})_{m\ge1}$, we prove the theorem. □

Proof of Theorem 5. $(\mathscr{G}_n)_{n\ge0}$ is a sequence of coverings of $\sigma(H_{\alpha,V})$ with diameter tends to 0. So

$$\dim_H \sigma(H_{\alpha,V}) \le s_*.$$

Now take any $\beta < s_*$, then $s_n > \beta$ for all n large enough, thus

$$\sum_{B\in\mathscr{G}_n} |B|^\beta > \sum_{B\in\mathscr{G}_n} |B|^{s_n} = 1.$$

Let μ_β be a Gibbs-like measure defined in Theorem 4. Then for any large k and each $B \in \mathscr{G}_k$ we have

$$\mu_\beta(B) \le \eta |B|^\beta.$$

Take $r > 0$ small and r-Moran covering of $\sigma(H_{\alpha,V})$, i.e.,

$$\mathscr{M} = \{B \in \mathscr{G}_n \ :\ n \ge 0, B \subset B^{-1} \in \mathscr{G}_{n-1}, |B^{-1}| > r, |B| \le r\}.$$

By Proposition A5 and Theorem 1, for any $B \in \mathscr{M}$,

$$|B| > \frac{r}{\xi^2(V+5)(M+2)^3}.$$

For any ball $B(x,r)$, letting $\mathscr{C} = \{B \in \mathscr{M} : B \cap B(x,r) \ne \emptyset\}$,

$$\sharp\mathscr{C} \le 3\xi^2(V+5)(M+2)^3.$$

Then,

$$\mu_\beta(B(x,r)) \le \sum_{B\in\mathscr{C}} \mu_\beta(B) \le \eta \sum_{B\in\mathscr{C}} |B|^\beta \le 3\eta\xi^2(V+5)(M+2)^3 r^\beta,$$

which implies $\dim_H E > \beta$. Hence, $\dim_H E \ge s_*$. □

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References

[1] J. Bellissard, B. Iochum, E. Scoppola and d. Testart, Spectral properties of one dimensional quasi-crystals, *Commun. Math. Phys.* **125**(1989), 527-543.

[2] D. Damanik, R. Killip, D. Lenz, Uniform spectral properties of one-dimensional quasicrystals, III. α-continuity, *Commun. Math. Phys.* **212**, (2000),191-204.

[3] D. Damanik, M. Embree, A. Gorodetski, and S. Tcheremchantsev, the fractal dimension of the spectrum of the Fibonacci Hamiltonian, *Commun. Math. Phys.***280:2**(2008), 499-516.

[4] D.-J. Feng, Z.-Y. Wen and J. Wu, *Some dimensional results for homogeneous Moran sets*, *Science of China*(Series A), **40:5**(1997), 475-482.

[5] S. Hua, H. Rao, Z.-Y. Wen and J. Wu, *On the structures and dimensions of Moran sets*, *Science of China*(Series A), **43:8**(1997), 836-852.

[6] D. Lenz, Singular Spectrum of Lebesgue Measure Zero for One-Dimensional Quasicrystals, *Commun. Math. Phys.* **227**(2002), 119–130.

[7] Q.H. Liu, J. Peyrière and Z.Y. Wen, Dimension of the spectrum of one-dimensional discrete Schrodinger operators with Sturmian potentials, *Comptes Randus Mathematique* **345:12**(2007), 667–672.

[8] Q.H. Liu, B. Tan, Z.X. Wen and J. Wu, Measure zero spectrum of a calss of Schrödinger operator, *J. Statist. Phys.*, **106:3-4** (2002), 681–691.

[9] Q.H. Liu, Z.Y. Wen, Hausdorff dimension of spectrum of one-dimensional Schrödinger operator with Sturmian potentials, *Potential Analysis* **20:1**(2004), 33–59.

[10] 7. Q.H. Liu, Z.Y. Wen, On dimensions of multitype Moran sets, *Math. Proc. Camb. Phyl. Soc.* **139:3**(2005), 541–553.

[11] J.H. Ma, H. Rao, Z.Y. Wen, *Dimensions of Cookie-cutter-like sets*, *Science of China*(Series A), **44:11**(2001), 1400-1412.

[12] L. Raymond, *A constructive gap labelling for the discrete schrödinger operater on a quasiperiodic chain.*(Preprint,1997)

[13] D. Schechtman, I. Blech, D. Gratias, and J. Cahn, Metallic phase with long-range orientational order and no translational symmetry, *Phys. Rev. Lett.* **53**(1984),1951–1954.

[14] A. Sütö, Schrödinger difference equation with deterministic ergodic potentials, *Beyond Quasicrystals*, Eds F.Alex and D.Gratias, Springer-Verlag, 1994.

[15] M. Toda, *Theory of Nonlinear Lattices*, Number 20 in Solid-State Sciences, Springer-Verlag, second enlarged edition, 1989. Chap. 4.